MAL 180 Discrete Mathematics

Major Test (November 2009)

Time: 2 Hours | Max. Marks: 50

Attempt any three questions in addition to questions 1 and 2.

1. (a) A player randomly chooses one of the coins A, B where A is biased with probability $P(Heads) = \frac{3}{4}$ and B is biased with probability $P(Heads) = \frac{1}{4}$. When the coin is tossed twice, find the probability that the player obtains (i) two heads, (ii) one head.

   If as an alternate strategy, a single unbiased coin can be tossed twice $[P(Heads) = \frac{1}{2}]$ find the strategy to maximize the probability of at least one head (in two tosses).

   (b) Using Kolmogorov's definition of probability, prove

$$p(A_1 \cap A_2) = p(A_1) + p(A_2) - p(A_1 \cup A_2)$$

   and hence derive the Bon feroni inequality

$$p(A_1 \cap \ldots \cap A_n) \geq \sum_{i=1}^{n} p(A_i) - (n-1)$$

2. Define two binary operations $\oplus$ and $\odot$ on $\mathbb{Z}$ by

$$\begin{aligned} a \oplus b = &\ a + b - 1 \\ a \odot b = &\ a + b - ab \quad \forall\, a, b \in \mathbb{Z} \end{aligned}$$

   Show that $(\mathbb{Z}, \oplus, \odot)$ is a ring. Further by exhibiting an isomorphism, show that this ring is isomorphic to the ring $(\mathbb{Z}, +, .)$ [[ring of integers under usual addition and multiplication]].

3. (a) A student takes a multiple choice test consisting of two problems, one having 3 possible answers and the other having 5 possible answers. If X stands for the right answers of the student, what are the mean and variance of X?

   (b) Let n papers be given for sale and let X be the number of papers sold follow the Poisson distribution with parameter λ. If Y stands for the net gain of the dealer where for each sold paper the gain is a cents while for each unsold paper the loss is b cents, find $E(Y)$ and the number of papers the dealer should get to maximize the profit.

4. (a) Define the chromial [[=chromatic polynomial]] of a graph and prove that a graph of order n is a tree *iff* its chromial is $x(x-1)^{n-1}$

   (b) Show that y_k, the expected number of occupied cells when k balls are at random placed into n cells, satisfies the recurrence relation $y_{k+1} = y_k(1 - \frac{1}{n}) + 1$ and solve it to get y_k.

5. (a) Prove that the group $(\mathbb{Q}, +)$[[rational numbers under usual addition]] is not a cyclic group.

   (b) Show that $\mathbb{Q}[\sqrt{2}\,] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is a subfield of the field $\mathbb{R}$

6. (a) Let $f(x) = x^5 + x^4 + 2x^2 + 3$, $g(x) = x^2 + 4x + 1 \in \mathbb{Z}_5[x]$. Find $q(x), r(x) \in \mathbb{Z}_5[x]$ such that $f(x) = q(x)g(x) + r(x)$, where either $r(x) = 0$ or $0 \leq deg(r(x)) < deg(g(x))$.

   (b) Find the complete factorization of $f(x) = 1 + x + 2x^2 + x^3 + x^4$ over $\mathbb{Z}_3$.

SMILE